Geometry & Topology Monographs 10 (2007) 333–346 **333**
arXiv version: fonts, pagination and layout may vary from GTM published version

Interactions of strings and equivariant homology theories

SHINGO OKUYAMA
KAZUHISA SHIMAKAWA

We introduce the notion of the space of parallel strings with partially summable labels, which can be viewed as a geometrically constructed group completion of the space of particles with labels. We utilize this to construct a machinery which produces equivariant generalized homology theories from such simple and abundant data as partial monoids.

55N20, 55N91; 55P47

1 Introduction

In [6] we attached to any pair of a Euclidean space V and a partial abelian monoid M a space $C(V,M)$ whose points are pairs consisting of a finite subset c of V and a map $a\colon c \to M$, but (c,a) is identified with (c',a') if $c \subset c'$, $a'|c = a$, and $a'(v) = 0$ for $v \notin c$. Any such pair (c,a) can be identified with the set consisting of "labeled particles" $(v, a(v))$, $v \in c$. Suppose V is an orthogonal G–module for some finite group G and M admits a G–action compatible with partial sum operations. Then $C(V,M)$ is a G–space with respect to the G–action

$$g(c,a) = (gc, gag^{-1}), \quad g \in G,\ (c,a) \in C(V,M).$$

Let $I(\mathbb{R})$ be the space of finite disjoint unions of bounded intervals in the real line. Then $I(\mathbb{R})$ is a partial abelian monoid with partial sum operation given by superimposition. Let us denote $I(V,M) = C(V, I(\mathbb{R}) \wedge M)$ for any partial abelian monoid with G–action M. Observe that under the correspondence

$$a\colon\ c \to I(\mathbb{R}) \quad \mapsto \quad \bigcup\nolimits_{v\in c}\{v\} \times a(v) \subset V \times \mathbb{R}$$

any map from a finite subset of V to $I(\mathbb{R})$ can be identified with a finite disjoint union of bounded subsets of the form $\{v\} \times J \subset V \times \mathbb{R}$, where J is a bounded interval. We call such $\{v\} \times J$ a *string* over v. Thus $I(V,M)$ can be regarded as the space consisting of finite sets of pairwise disjoint labeled strings whose members over the same point in V has the same label in M.

Published: 18 April 2007 DOI: 10.2140/gtm.2007.10.333

The aim of this paper is to show that if V is sufficiently large then there is a G–equivariant group completion map $C(V,M) \to I(V,M)$ and also that the correspondence $X \mapsto \pi_n I(V, X\wedge M)$, $n \geq 0$, extends to an $RO(G)$–graded generalized homology theory.

To state the precise results, let $\mathrm{Top}(G)$ be the category of all pointed G–spaces and all pointed maps with G acting on maps by conjugation. In [5] we have shown that any G–equivariant continuous functor $T\colon\ \mathrm{Top}(G) \to \mathrm{Top}(G)$ such that $T(*) = *$ is associated with pairings $X \wedge TY \to T(X\wedge Y)$, $TX \wedge Y \to T(X \wedge Y)$ natural in both X and Y. Therefore, T preserves G–homotopies and there is a natural transformation $S^W \wedge T(X) \to T(S^W \wedge X)$ for any orthogonal G–module W, where S^W is the one point compactification of W.

Suppose V is linearly and equivariantly isometric to the direct product of countably many copies of the regular representation of G over the real number fields. Such a G–module V is called a G–universe. Now the main results can be stated as follows.

Theorem 1.1 *There is a diagram consisting of maps of Hopf G–spaces*

$$C(V,M) \xleftarrow{\lambda} I_+(V,M) \xrightarrow{\rho} I(V,M)$$

satisfying the conditions below.

(1) *λ is a G–homotopy equivalence.*

(2) *ρ is an equivariant group completion, that is to say, it restricts to a group completion map $I_+(V,M)^H \to I(V,M)^H$ for every subgroup H of G.*

Theorem 1.2 *The correspondence $X \to I(V, X\wedge M)$ is a G–equivariant continuous functor of $\mathrm{Top}(G)$ into itself and we have the following:*

(1) *For any orthogonal G–module W the natural map*

$$I(V, X\wedge M) \to \Omega^W I(V, \Sigma^W X \wedge M)$$

adjoint to $S^V \wedge I(V, X\wedge M) \to I(V, S^W \wedge X \wedge M)$ is a weak G–equivalence.

(2) *There exists an $RO(G)$–graded homology theory $h^G_\bullet(-)$ such that*

$$h^G_n(X) = \pi_n I(V, X\wedge M)^G$$

holds for any X and $n \geq 0$.

These theorems enable us to construct equivariant generalizations of several popular homology theories. For example, consider the simplest case $M = S^0$. Then $C(V,X)$ is the usual configuration space, and hence its group completion $I(V,X)$ is weakly

G–equivalent to the equivariant infinite loop space $\Omega^V\Sigma^V X$ by [1, Theorem (1.18)]. Thus we obtain the G–equivariant stable homotopy theory in this case. On the other hand, if we take arbitrary positive integers as labels then we obtain an $RO(G)$–graded homology theory extending the ordinary homology $\widetilde{H}_n(X/G,\mathbb{Z})$. (Compare Lewis, May and McClure [2].) K–theory type examples also occur from our method, which will be discussed in a future paper.

2 Partial abelian monoids with G–action

Definition 2.1 A pointed G–space M is called a partial abelian monoid with G–action, or *G–partial monoid* for short, if for every $n \geq 0$ there are G–invariant subsets M_n of M^n and G–maps

$$M_n \to M, \quad (a_1,\dots,a_n) \mapsto a_1 + \cdots + a_n$$

satisfying the conditions below.

(1) $M_0 \to M$ is the inclusion of the basepoint 0 of M.

(2) $M_1 \to M$ is the identity of M.

(3) Let $J_1, \cdots, J_r$ be disjoint subsets of $\{1,\dots,n\}$ such that $J_1 \cup \cdots \cup J_r = \{1,\dots,n\}$, and let $(a_1,\dots,a_n)$ be an element of M^n such that $(a_j)_{j\in J_k}$ belongs to M_{n_k}, where n_k is the cardinality of J_k. Then $(a_1,\dots,a_n) \in M_n$ if and only if $\left(\sum_{j\in J_1} a_j,\dots,\sum_{j\in J_r} a_j\right) \in M_r$, and we have

$$a_1 + \cdots + a_n = \textstyle\sum_{j\in J_1} a_j + \cdots + \sum_{j\in J_r} a_j$$

if either side of the equation makes sense.

Among the examples we have the following:

(1) Let M be a G–invariant subset of a topological abelian group on which G acts through group homomorphisms. Suppose M contains the unit 0. Then M is a G–partial monoid with respect to the subsets

$$M_n = \{(a_1,\dots,a_n) \in M^n \mid a_1 + \cdots + a_n \in M\}.$$

More generally, any G–invariant subset of a G–partial monoid that contains 0 is again a G–partial monoid.

(2) Any pointed G–space X is a G–partial monoid with respect to folding maps $X_n = X \vee \cdots \vee X \to X$. In fact, this is a special case of the previous example, as X is a G–invariant subset of the infinite symmetric product $\mathrm{SP}^\infty X$.

(3) Let V be an infinite dimensional real inner product space on which G acts through linear isometries. Then the Grassmannian $\mathrm{Gr}(V)$ of finite-dimensional subspaces of V is a G–partial monoid with respect to the inner direct sum operation $\mathrm{Gr}(V)_n \to \mathrm{Gr}(V)$, where $\mathrm{Gr}(V)_n$ is defined to be the subset consisting of those $(W_1, \dots, W_n)$ such that $W_i \perp W_j$ if $i \neq j$.

Definition 2.2 For given G–partial monoids M and N, their smash product $M \wedge N$ is a G–partial monoid such that $(M \wedge N)_n$ is the subset consisting of those n–tuples that can be summed up to an element of $M \wedge N$ by using the distributivity relations:

$$\begin{aligned} c_1 \wedge d + \cdots + c_k \wedge d &= (c_1 + \cdots + c_k) \wedge d, \qquad (c_1, \dots, c_k) \in M_k \\ c \wedge d_1 + \cdots + c \wedge d_l &= c \wedge (d_1 + \cdots + d_l), \qquad (d_1, \dots, d_l) \in N_l \end{aligned}$$

Example 2.3 If X is a pointed G–space and M is a G–partial monoid, then $X \wedge M$ is a G–partial monoid such that

$$(X \wedge M)_n = X \wedge M_n$$

holds for every $n \geq 0$.

For any orthogonal G–module V, the labeled configuration space $C(V, M)$ is a G–partial monoid with respect to the partial sum operations

$$C(V, M)_n \to C(V, M), \quad ((c_1, a_1), \dots, (c_n, a_n)) \mapsto \left(\bigcup c_i, \bigcup a_i\right).$$

Here $C(V, M)_n$ consists of those n–tuples $((c_i, a_i)) \in C(V, M)^n$ such that for every $x \in \bigcup c_i$ the sum $\sum_{i \in \Lambda(x)} a_i(x)$ exists, where $\Lambda(x) = \{i \mid x \in c_i\}$, and $\bigcup a_i$ denotes the map $x \mapsto \sum_{i \in \Lambda(x)} a_i(x)$. Moreover, if V is a G–universe then $C(V, M)$ is a homotopy associative and homotopy commutative Hopf G–space. To see this, let us consider the functor

$$P \mapsto A(P) = C(V, P \wedge M)$$

from finite pointed sets to pointed G–spaces. For each $p \in P$, let δ_p be the pointed map $P \to \mathbf{1} = \{0, 1\}$ such that $\delta_p^{-1}(1) = \{p\}$ if p is not the basepoint of P, and let δ_p be the constant map if p is the basepoint. Then the G–map

$$\delta\colon\ A(P) \to \mathrm{Map}_0(P, A(\mathbf{1})), \quad a \mapsto (p \mapsto A(\delta_p)(a))$$

has a G–homotopy inverse $\psi\colon\ \mathrm{Map}_0(P, A(\mathbf{1})) \to A(P)$ defined as follows.

Since V is a G–universe, there exist an embedding of $P - \{0\}$ into V^G and a G–linear isometry $V \times V \to V$. Hence we can construct a G–equivariant embedding of

$(P-\{0\})\times V$ into V. For any $f\in \mathrm{Map}_0(P,A(\mathbf{1}))$ let us write $f(p)=(c(p),a(p))$ and put $\psi(f)=(\hat{c},\hat{a})\in A(P)$, where $\hat{c}$ is the image of

$$\bigcup_{p\in P-\{0\}}\{p\}\times c(p)$$

under the G–equivariant embedding $(P-\{0\})\times V\to V$ and $\hat{a}\colon\ \hat{c}\to P\wedge M$ is induced by the composite maps

$$c(p)\xrightarrow{a(p)} M=\mathbf{1}\wedge M\xrightarrow{\iota_p\wedge 1} P\wedge M$$

where ι_p is a pointed map $\mathbf{1}\to P$ such that $\iota_p(1)=p$.

Therefore, A is a G–equivariant Γ–space in the sense of Segal. Hence the following proposition holds.

Proposition 2.4 *$C(V,M)$ is a homotopy associative and homotopy commutative Hopf G–space with unit $\emptyset\in C(V,M)^G$.*

Note that Hopf G–space multiplication μ of $C(V,M)$ is given by the composite

$$C(V,M)^2\underset{\simeq}{\xrightarrow{\psi}} C(V,M\vee M)\xrightarrow{\nabla_*} C(V,M)$$

where ∇_* is induced by the folding map $M\vee M\to M$.

Definition 2.5 A G–partial monoid M is *homotopically invertible* if there exist a map of G–partial monoids $\tau\colon\ M\to M$, called a *homotopy inversion*, and a G–homotopy $h_t\colon\ M\to M^2$ $(0\le t\le 1)$ satisfying the conditions below.

(1) For every $t\in[0,1]$, h_t is a map of G–partial monoid.

(2) $h_0=(1,\tau)$, ie we have $h_0(a)=(a,\tau(a))$ for any $a\in M$.

(3) h_1 factors through a map $h_1'\colon\ M\to M_2$ and the composite $M\xrightarrow{h_1'} M_2\xrightarrow{\Sigma} M$ is G–homotopic through maps of G–partial monoids to the constant map.

Proposition 2.6 *If V is a G–universe and if M is homotopically invertible then $C(V,M)$ is a grouplike Hopf G–space.*

Proof Let $\tau_*\colon\ C(V,M)\to C(V,M)$ be the map induced by the homotopy inversion of M. To see that $C(V,M)$ is grouplike, it suffices to show that the composite

$$C(V,M)\xrightarrow{(1,\tau_*)} C(V,M)^2\xrightarrow{\mu} C(V,M)$$

is G–homotopic to the constant map with value $\emptyset$. Let us regard $M \times M$ as a G–partial monoid such that $(M \times M)_n = M_n \times M_n$ for $n \geq 0$. Then we have a diagram of pointed G–spaces

$$\begin{tikzcd}
C(V,M) \arrow[r, "(1,\tau)_*"] \arrow[d, "h'_{1*}"'] & C(V,M^2) \arrow[r, "(p_{1*},p_{2*})"] & C(V,M)^2 \\
C(V,M_2) \arrow[r, equal] & C(V,M_2) \arrow[u] \arrow[d, "\Sigma_*"'] & C(V,M \vee M) \arrow[l] \arrow[u, "\simeq"', "\delta"] \arrow[d, "\nabla_*"] \\
 & C(V,M) \arrow[r, equal] & C(V,M)
\end{tikzcd}$$

in which p_{1*} and p_{2*} are induced by the projections $M^2 \to M$ onto the first and the second factors, respectively, and unnamed arrows are induced by the inclusions of G–partial monoids. Clearly, the right hand side squares are commutative, and the upper left square commutes up to G–homotopy. Since δ has a G–homotopy inverse ψ and since ψ restricts to a G–homotopy inverse to the map $C(V, M \vee M) \to C(V, M^2)$ induced by the inclusion $M \vee M \to M^2$, all the arrows constituting the upper right square are G–homotopy equivalences. Thus we have

$$\mu(1, \tau_*) = \nabla_* \psi(p_{1*}, p_{2*})(1, \tau)_* \simeq \Sigma_* h'_{1*} \simeq \emptyset. \qquad \square$$

3 The space of strings with labels

As usual, the symbols $[a,b]$, $[a,b)$, $(a,b]$, (a,b) represent bounded intervals in the real line, and $b-a$ is called the length of the interval. The space of intervals $I(\mathbb{R})$ consists of those unions $P = J_1 \cup \cdots \cup J_r$ of finite number of pairwise disjoint bounded intervals. It is topologized in such a way that such operations as isotopy moves, concatenation of two disjoint intervals that have a connected union (eg $[a,c) \cup [c,b] = [a,b]$), and deletion of a half-open interval when its length tends to 0 are all continuous. Let $I(\mathbb{R})_n$ be the subset of $I(\mathbb{R})^n$ consisting of those n–tuples $(P_1, \ldots, P_n)$ that are pairwise disjoint. Then $I(\mathbb{R})$ is a partial abelian monoid with respect to these $I(\mathbb{R})_n$ and maps

$$I(\mathbb{R})_n \to I(\mathbb{R}), \quad (P_1, \ldots, P_n) \mapsto P_1 \cup \cdots \cup P_n.$$

Details are given in Okuyama [3], where $I(\mathbb{R})$ is denoted by $I_1(S^0)$.

Lemma 3.1 *$I(\mathbb{R})$ is a homotopically invertible partial abelian monoid.*

Proof Given a bounded interval J, let τJ denote the complement of the boundary of $-J$ in its closure. To be more explicit, we put

$$\tau[a,b]=(-b,-a),\quad \tau(a,b)=[-b,-a],\quad \tau[a,b)=[-b,-a),\quad \tau(a,b]=(-b,-a].$$

Then the correspondence $J\mapsto \tau J$ extends to an involution τ of $I(\mathbb{R})$

$$J_1\cup\cdots\cup J_r\mapsto \tau J_r\cup\cdots\cup\tau J_1.$$

Let $\alpha\colon\thinspace \mathbb{R}\to(0,1)$ be an order preserving homeomorphism and let

$$\alpha_t(s)=(1-t)s+t\alpha(s)$$

for $t\in[0,1]$ and $s\in\mathbb{R}$. Since $\alpha_t\colon\thinspace\mathbb{R}\to\mathbb{R}$ is an embedding, it induces a map of partial monoids $I(\alpha_t)\colon\thinspace I(\mathbb{R})\to I(\mathbb{R})$ for every t, and hence we can define a homotopy $h_t\colon\thinspace I(\mathbb{R})\to I(\mathbb{R})^2$ by

$$h_t(P)=(I(\alpha_t)(P),\tau I(\alpha_t)(P)).$$

Clearly, h_t is a map of partial monoids and we have $h_0=(1,\tau)$ because $I(\alpha_0)$ is the identity. On the other hand, h_1 maps $I(\mathbb{R})$ into $I(\mathbb{R})_2$ because $I(\alpha)(P)$ is contained in $(0,1)$ and hence is disjoint from $\tau I(\alpha)(P)\subset(-1,0)$. Finally, we can define a homotopy $\Sigma h_1\simeq\emptyset$ by moving $I(\alpha)(P)$ to negative direction and $\tau I(\alpha)(P)$ to positive direction, simultaneously, so that the strings J in $I(\alpha)(P)$ meet with τJ at the origin and the resulting half-open intervals eventually vanish. □

Let $I(\mathbb{R})_+$ be the subset of $I(\mathbb{R})$ consisting of those $J_1\cup\cdots\cup J_r$ such that every J_i is a closed interval. Clearly, $I(\mathbb{R})_+$ is a partial submonoid of $I(\mathbb{R})$.

Definition 3.2 Given an orthogonal G–module V and a G–partial monoid M, let

$$I(V,M)=C(V,I(\mathbb{R})\wedge M),\quad I_+(V,M)=C(V,I(\mathbb{R})_+\wedge M).$$

For any G–partial monoid M, $I(\mathbb{R})\wedge M$ is a homotopically invertible G–partial monoid with homotopy inversion $\tau\wedge 1$. Thus we have the following proposition.

Proposition 3.3 *If V is a G–universe then $I(V,M)$ is grouplike for any M.*

4 Proof of Theorem 1.1

To establish a relation between $I(V,M)$ and $C(V,M)$, let us choose a linear embedding $e\colon\thinspace\mathbb{R}\to V^G$ and a G–linear isometry $l\colon\thinspace V\times V\to V$. Then we can define

$$\lambda\colon\thinspace I_+(V,M)\to C(V,M)$$

to be the map which sends a finite set of labeled strings $\{(\{v_i\} \times J_i, a_i)\}$ to the set of labeled particles $\{(l(v_i, e(\hat{J}_i)), a_i)\}$, where $\hat{J}_i$ is the middle point of the closed interval J_i. Note that $(v_i, e(\hat{J}_i))$ are pairwise distinct, hence so are $l(v_i, e(\hat{J}_i))$.

Proposition 4.1 *$\lambda\colon I_+(V,M) \to C(V,M)$ is a G–homotopy equivalence of Hopf G–spaces.*

Proof Since λ is natural with respect to M, it extends to a map of G–equivariant Γ–spaces. This, of course, implies that λ is a map of Hopf G–spaces.

To see that λ is a G–homotopy equivalence, let $\gamma\colon C(V,M) \to I_+(V,M)$ be a pointed G–map which sends a finite set of labeled particles $\{(v_i, a_i)\}$ to the set of labeled strings $\{(\{v_i\} \times [-1,1], a_i)\}$. Then we have

$$\gamma\lambda(\{(\{v_i\} \times J_i, a_i)\}) = \{(\{l(v_i, e(\hat{J}_i))\} \times [-1,1], a_i)\}$$

and we can define a G–homotopy $\gamma\lambda \simeq 1$ by

$$(\gamma\lambda)_t(\{(\{v_i\} \times J_i, a_i)\}) = \begin{cases} \{(\{l(v_i, e_{2t}(\hat{J}_i))\} \times \mathcal{I}_{2t}(J_i), a_i)\}, & 0 \le t \le 1/2 \\ \{(\{l_{2t-1}(v_i)\} \times J_i, a_i)\}, & 1/2 \le t \le 1 \end{cases}$$

where

1. $e_t\colon \mathbb{R} \to V^G$ is a linear map $s \mapsto (1-t)e(s)$.
2. If $J = [a,b]$ then $\mathcal{I}_t(J) = [ta-(1-t), tb+(1-t)]$. Thus $\{\mathcal{I}_t(J)\}$ is a continuous family of closed intervals such that $\mathcal{I}_0(J) = [-1,1]$ and $\mathcal{I}_1(J) = J$.
3. $\{l_t\}$ is a continuous family of G–linear isometries $V \to V$ such that $l_0(v) = l(v,0)$ and l_1 is the identity of V. (Such a family certainly exists because the space of G–linear isometries $V \to V$ is contractible if V is a G–universe.)

On the other hand, we can define a G–homotopy $\lambda\gamma \simeq 1$ by

$$(\lambda\gamma)_t(\{(v_i, a_i)\}) = \{(l_t(v_i), a_i)\}. \qquad \square$$

Now let $\rho\colon I_+(V,M) \to I(V,M)$ be the map induced by the inclusion $I(\mathbb{R})_+ \subset I(\mathbb{R})$. To complete the proof of Theorem 1.1, we need to show that

$$\rho^H\colon I_+(V,M)^H \to I(V,M)^H \tag{4–1}$$

is a group completion for every subgroup H of G. Since V is an H–universe for any subgroup H of G, we need only consider the case $H = G$. But then we have:

Lemma 4.2 *$I_+(V,M)^G \to I(V,M)^G$ is a group completion for a G–partial monoid M if so is $I_+(\mathbb{R}^\infty, M) \to I(\mathbb{R}^\infty, M)$ for all partial abelian monoids M.*

Proof Let $\mathcal{F}$ be a family of orbit types and let $C(V,M)_{\mathcal{F}}$ be the subspace of $C(V,M)$ consisting of those $(c,v)\in C(V,M)$ such that $c\in V_{\mathcal{F}}$, where $V_{\mathcal{F}}=\{v\in V|G_v\in\mathcal{F}\}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families such that $\mathcal{F}_1\subset\mathcal{F}_2$ and $\mathcal{F}_2-\mathcal{F}_1$ consists of only one conjugacy class (H) then we have a fibration sequence

$$C(V,M)^G_{\mathcal{F}_1}\to C(V,M)^G_{\mathcal{F}_2}\to C(V,M)^G_{(H)}.$$

Therefore, we see that $I_+(V,M)^G\to I(V,M)^G$ is a group completion if and only if so are $I_+(V,M)^G_{(H)}\to I(V,M)^G_{(H)}$, by arguing as in Section 6 of Caruso and Waner [1]. But we have

$$C(V,M)^G_{(H)}\cong C(V^H,M^H)^{NH}_{(H)}\simeq C(\mathbb{R}^\infty,EW(H)\wedge_{W(H)}M^H).$$

It follows that $I_+(V,M)^G\to I(V,M)^G$ is a group completion if so are

$$I_+(\mathbb{R}^\infty,EW(H)\wedge_{W(H)}M^H)\to I(\mathbb{R}^\infty,EW(H)\wedge_{W(H)}M^H).\qquad\square$$

In order to prove the lemma in the non equivariant case we need a CW–monoid replacement for $I(\mathbb{R}^\infty,M)$. For any M let $|S_\bullet M|$ be the realization of the total singular complex of M regarded as a partial abelian monoid such that

$$|S_\bullet M|_n=|S_\bullet M_n|\subset|S_\bullet M|^n\quad(n\geq 0).$$

Let $D(M)$ be the classifying space of the permutative category $\mathcal{Q}(|S_\bullet M|)$ whose space of objects is $\coprod_{p\geq 0}|S_\bullet M|^p$ and whose morphisms from $(a_i)\in|S_\bullet M|^p$ to $(b_j)\in|S_\bullet M|^q$ are given by a map of finite sets $\theta\colon\ \{1,\dots,p\}\to\{1,\dots,q\}$ such that $b_j=\sum_{i\in\theta^{-1}(j)}a_i$ hold. Then $D(M)$ is a CW–monoid since it is homeomorphic to the realization of the diagonal simplicial set $[n]\mapsto N_n\mathcal{Q}(S_nM)$. Moreover, there is a natural weak equivalence of Hopf spaces $\Phi\colon\ D(M)\to C(\mathbb{R}^\infty,M)$. (For details, see Shimakawa [7, Section 2.4].) Thus to prove Lemma 4.2 we need only show the following

Proposition 4.3 *The natural map $D(I(\mathbb{R})_+\wedge M)\to D(I(\mathbb{R})\wedge M)$ induced by the inclusion $I(\mathbb{R})_+\subset I(\mathbb{R})$ is a group completion.*

The rest of this section is devoted to the proof of this proposition.

Given a map of topological monoids $f\colon\ D\to D'$ let $B(D,D')$ denote the realization of the category $\mathcal{B}(D,D')$ whose space of objects is D' and whose space of morphisms is the product $D\times D'$, where $(d,d')\in D\times D'$ is regarded as a morphism from d' to $f(d)\cdot d'$. Then there is a sequence of maps

$$D'=B(0,D')\to B(D,D')\to B(D,0)=BD$$

induced by the maps $0 \to D$ and $D' \to 0$ respectively. Observe that BD is the standard classifying space of the monoid D and $B(D,D)$ is contractible when f is the identity.

In particular, let us take $D = D(I(\mathbb{R})_+ \wedge M)$ and $D' = D(I(\mathbb{R}) \wedge M)$, and let $i\colon D \to D'$ be the monoid map induced by the inclusion $I(\mathbb{R})_+ \to I(\mathbb{R})$. Then there is a commutative diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d, "i"'] & B(D,D) \arrow[r] \arrow[d, "B(1,i)"] & BD \arrow[d, equal] \\ D' \arrow[r] & B(D,D') \arrow[r] & BD \end{tikzcd} \tag{4–2}$$

in which the upper and the lower sequences are associated with the identity and the inclusion $i\colon D \to D'$, respectively.

Lemma 4.4 *The natural map $D \to \Omega BD$ is a group completion.*

This follows from the fact that D is a homotopy commutative monoid.

Lemma 4.5 *The lower sequence in the diagram (4–2) is a homotopy fibration sequence with contractible total space.*

Proposition 4.3 is deduced from this, because $D \to D'$ is equivalent to the group completion map $D \to \Omega BD$ under the equivalence $D' \simeq \Omega BD$.

Proof of Lemma 4.5 By Proposition 3.3, $D' = D(I(\mathbb{R}) \wedge M)$ is grouplike with homotopy inverse induced by the homotopy inversion $\tau \wedge 1$. Hence D acts on D' through homotopy equivalences, and the diagram

$$\begin{tikzcd} D' \arrow[r] \arrow[d] & B(D,D') \arrow[d] \\ 0 \arrow[r] & B(D,0) \end{tikzcd}$$

is homotopy cartesian by Proposition 1.6 of Segal [4]. This implies that the lower sequence in the diagram (4–2) is a homotopy fibration sequence.

It remains to prove that $B(D,D')$ is contractible. In [7], we proved this in the case where the partial monoid $X \wedge \pm M$ is strictly invertible and is generated by the elements of $X \wedge M$ and their inverses. But the argument given there still applies to the current case, once we make the following change in the notation.

Replace $X \wedge M$ and $X \wedge \pm M$ by $I_+(\mathbb{R}) \wedge M$ and $I(\mathbb{R}) \wedge M$, respectively, and for any $S = (P_j \wedge a_j) \in S_0(I(\mathbb{R}) \wedge M)^p$ put

$$S_+ = (P_j^+ \wedge a_j), \quad S_- = (P_j^- \wedge a_j), \quad \overline{S} = (\tau P_j \wedge a_j),$$

where P_j^+ and P_j^- are the unions of closed intervals and of open or half-open intervals contained in P_j, respectively. Note that we have $P_j = P_j^+ \bigcup P_j^-$ and $P_j^+ \in I_+(\mathbb{R})$. Also, for any S such that $S = S_-$ the path $[S] \to [\mathbf{0}^p]$ in $B(D, D')$ should be defined to be the composite

$$[S] \to [\overline{S}_+ \cdot S] \to [\overline{I(\alpha)_*(S)}_+ \cdot I(\alpha)_*(S)] \xrightarrow{\nabla} [\mathbf{0}^p]$$

where α is a homeomorphism $\mathbb{R} \cong (0,1)$, $I(\alpha)_*(P_j \wedge a_j) = (I(\alpha)(P_j) \wedge a_j)$, and ∇ is induced by the homotopy

$$\tau I(\alpha)(P_j)^+ \wedge a_j + I(\alpha)(P_j) \wedge a_j = (\tau I(\alpha)(P_j)^+ \bigcup I(\alpha)(P_j)) \wedge a_j \simeq \emptyset \wedge a_j = 0.$$

(Compare the proof of Lemma 3.1.) □

5 Proof of Theorem 1.2

By a simplicial pointed G–space we shall mean a simplicial object in the category of pointed G–spaces and basepoint preserving G–maps. If $X_\bullet$ is a simplicial pointed G–space then the basepoints of X_n form the simplicial set $*$. Let

$$\|X_\bullet\|' = \|X_\bullet\| / \|*\|.$$

Then the natural map $\|X_\bullet\| \to \|X_\bullet\|'$ is a G–homotopy equivalence, and the maps $\Delta^n \times X_n \to \|X_\bullet\|$ induce $\Delta^n_+ \wedge X_n \to \|X_\bullet\|'$.

Let T be a G–equivariant continuous functor $\mathrm{Top}(G) \to \mathrm{Top}(G)$. Then any simplicial pointed G–space $X_\bullet$ is associated with a G–map $\|T(X_\bullet)\|' \to T(\|X_\bullet\|')$ induced by the maps

$$\Delta^n \times T(X_n) \to \Delta^n_+ \wedge T(X_n) \to T(\Delta^n_+ \wedge X_n) \to T(\|X_\bullet\|') .$$

The next proposition plays a key role in the proof of Theorem 1.2.

Proposition 5.1 *Let $T\colon \mathrm{Top}(G) \to \mathrm{Top}(G)$ be a G–equivariant continuous functor. Suppose T satisfies the conditions below.*

(C1) $T(*) = *$.

(C2) For any simplicial pointed G–space $X_\bullet$ the natural map $\|T(X_\bullet)\|' \to T(\|X_\bullet\|')$ is a G–homotopy equivalence.

(C3) *For any X and Y the map $T(X \vee Y) \to T(X) \times T(Y)$ induced by the projections $X \vee Y \to X$ and $X \vee Y \to Y$ is a G–homotopy equivalence.*

(C4) *For any subgroup H the natural map $T(G/H_+ \wedge X) \to \mathrm{Map}_0(G/H_+, T(X))$, whose adjoint $G/H_+ \wedge T(G/H_+ \wedge X) \to T(X)$ is induced by the pairing $G/H_+ \wedge G/H_+ \wedge X \to X$ which sends (s, s, x) to x and (s, t, x) ($s \neq t$) to the basepoint of X, is a G–homotopy equivalence.*

Suppose further that $T(X)^H$ is grouplike for any X and any subgroup H of G. Then the following hold.

(1) *For any orthogonal G–module W the natural map $T(X) \to \Omega^W T(\Sigma^W X)$ adjoint to $S^W \wedge T(X) \to T(S^W \wedge X)$ is a weak G–homotopy equivalence.*

(2) *The correspondence $X \mapsto \left\{\pi_n T(X)^G\right\}$ is extendible to an $RO(G)$–graded equivariant homology theory defined on the category of pointed G–spaces.*

Proof For any pointed G–space X let $E(X) = \Omega T(\Sigma X)$. If T satisfies (C1), (C2) and (C3) then by the equivariant version of [6, Theorem 2.12] the natural map $T(X) \to E(X)$ is a G–equivariant group completion and the sequence

$$E(A) \to E(X) \to E(X \cup CA)$$

associated with a pair of pointed G–spaces (X, A) is a G–fibration sequence up to weak G–equivalence. But $T(X) \to E(X) = \Omega T(\Sigma X)$ is a weak G–equivalence because $T(X)^H$ is grouplike for any subgroup H. Hence

$$T(A) \to T(X) \to T(X \cup CA)$$

is a G–fibration sequence up to weak G–equivalence. Moreover, T preserves G–homotopies because it is a G–equivariant continuous functor. Therefore, the correspondence $X \mapsto \left\{\pi_n T(X)^G\right\}$ determines a $\mathbb{Z}$–graded equivariant homology theory.

Let Γ_G be the full subcategory of $\mathrm{Top}(G)$ consisting of finite pointed G–sets. To prove the assertions we need only show that the correspondence $S \mapsto T(S \wedge X)$ from Γ_G to $\mathrm{Top}(G)$ is a special Γ_G–space in the sense of [5]. But this follows from the conditions (C3) and (C4). □

Now let $T(X) = I(V, X \wedge M)$. We shall show that T satisfies the conditions (C1), (C2), (C3) and (C4). This of course proves Theorem 1.2.

It is obvious that (C1) holds. (C2) is proved by the argument similar to the one used in the proof of [6, Theorem 3.2]. To prove (C3) let us define

$$T(X) \times T(Y) \to T(X \vee Y)$$

to be the composite

$$I(V, X\wedge M)\times I(V, Y\wedge M)\xrightarrow{(i_*, j_*)} I(V,(X\vee Y)\wedge M)^2\xrightarrow{\mu} I(V,(X\vee Y)\wedge M)$$

where i_* and j_* are induced by the inclusions of X and Y into $X\vee Y$, respectively, and μ is the multiplication of the Hopf G–space $I(V,(X\vee Y)\wedge M)$. By using the fact that the space of G–linear isometries of V is contractible one can show that the map above gives a G–homotopy inverse to $T(X\vee Y)\to T(X)\times T(Y)$. Finally, to prove (C4) let us choose a G–embedding $G/H\to V$ and a G–linear isometry $l\colon V\times V\to V$. Then we can construct a G–homotopy inverse to the natural map $T(G/H_+\wedge X)\to \mathrm{Map}_0(G/H_+, T(X))$ by the following procedure:

(1) For given $f\colon G/H_+\to T(X)$ let us write

$$f(gH)=(c(gH), P(gH)\wedge a(gH))$$

where $c(gH)\subset V$, $P(gH)\colon c(gH)\to I(\mathbb{R})$ and $a(gH)\colon c(gH)\to X\wedge M$.

(2) Let $\tilde{c}$ be the image of the union $\bigcup\{gH\}\times c(gH)$ under the embedding

$$\iota\colon G/H\times V\subset V\times V\xrightarrow{l} V.$$

(3) Define $\tilde{a}\colon \tilde{c}\to I(\mathbb{R})\wedge G/H_+\wedge X\wedge M$ by

$$\tilde{a}(\iota(gH,\xi))=P(gH)\wedge gH\wedge a(gH)(\xi),\quad \xi\in c(gH).$$

(4) Define $\rho\colon \mathrm{Map}(G/H, T(X))\to T(G/H_+\wedge X)$ by $\rho(f)=(\tilde{c},\tilde{a})$.

That ρ gives a G–homotopy inverse to $T(G/H_+\wedge X)\to \mathrm{Map}(G/H, T(X))$ follows, again, from the contractibility of the space of G–linear isometries of V.

References

[1] **J Caruso**, **S Waner**, *An approximation theorem for equivariant loop spaces in the compact Lie case*, Pacific J. Math. 117 (1985) 27–49 MR777436

[2] **G Lewis**, **J P May**, **J McClure**, *Ordinary $RO(G)$–graded cohomology*, Bull. Amer. Math. Soc. (N.S.) 4 (1981) 208–212 MR598689

[3] **S Okuyama**, *The space of intervals in a Euclidean space*, Algebr. Geom. Topol. 5 (2005) 1555–1572 MR2186109

[4] **G Segal**, *Categories and cohomology theories*, Topology 13 (1974) 293–312 MR0353298

[5] **K Shimakawa**, *Infinite loop G–spaces associated to monoidal G–graded categories*, Publ. Res. Inst. Math. Sci. 25 (1989) 239–262 MR1003787

[6] **K Shimakawa**, *Configuration spaces with partially summable labels and homology theories*, Math. J. Okayama Univ. 43 (2001) 43–72 MR1913872

[7] **K Shimakawa**, *Labeled configuration spaces and group completions*, Forum Math. to appear (2006)

Takuma National College of Technology
Kagawa 769-1192, Japan

Takuma National College of Technology
Kagawa 769-1192, Japan

okuyama@dc.takuma-ct.ac.jp, kazu@math.okayama-u.ac.jp

Received: 13 September 2004 Revised: 13 May 2006